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Research Article

Asymptotic Behavior of a Periodic Diffusion System

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We study the asymptotic behavior of the nonnegative solutions of a periodic reaction diffusion system. By obtaining a priori upper bound of the nonnegative periodic solutions of the corresponding periodic diffusion system, we establish the existence of the maximum periodic solution and the asymptotic boundedness of the nonnegative solutions of the initial boundary value problem.

1. Introduction

In this paper, we consider the following periodic reaction diffusion system:

$$\frac{\partial u}{\partial t} = \Delta u^{m_1} + b_1 u^{\alpha_1} v^{\beta_1}, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta u^{m_2} + b_2 u^{\alpha_2} v^{\beta_2}, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.2}$$

with initial boundary conditions

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.4}$$

where $m_1, m_2 > 1$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 1$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, $b_1 = b_1(x,t)$ and $b_2 = b_2(x,t)$ are nonnegative continuous functions and of T-periodic $(T > 0)$ with respect to t, and u_0 and v_0 are nonnegative bounded smooth functions.

In dynamics of biological groups ([1, 2]), the system (1.1)-(1.2) was used to describe the interaction of two biological groups without self-limiting, where the diffusion terms reflect that the speed of the diffusion is slow. In addition, the system (1.1)-(1.2) can also be used to describe diffusion processes of heat and burning in mixed media with nonlinear conductivity and volume release, where the functions u, v can be treated as temperatures of interacting components in the combustible mixture [3].

For case of $m_1 = m_2 = 1$, we get the classical reaction diffusion system of Fujita type

$$\frac{\partial u}{\partial t} = \Delta u + u^{\alpha_1} v^{\beta_1}, \qquad \frac{\partial v}{\partial t} = \Delta v + u^{\alpha_2} v^{\beta_2}. \tag{1.5}$$

This type reaction diffusion system (1.5) models such as heat propagations in a two-component combustible mixture [4], chemical processes [5], and interaction of two biological groups without self-limiting [6, 7]. The problem about system (1.5) includes global existence and global existence numbers, blow-up, blow-up rates, blow-up sets, and uniqueness of weak solutions (see [8–10] and references therein).

In this paper, we will work on the diffusion system (1.1)-(1.2); for results about single equation, see [11–16] and so on. In the past two decades, the system (1.1)-(1.2) has been deeply investigated by many authors, and there have been much excellent works on the existence, uniqueness, regularity and some other qualitative properties of the weak solutions of the initial boundary value problem (see [17–22] and references therein). Maddalena [20] especially, established the existence and uniqueness of the solutions of the initial boundary value problem (1.1)–(1.4), and Wang [22] established the existence of the nonnegative nontrivial periodic solutions of the periodic boundary value problem (1.1)–(1.3) when $m_i > 1$, $\alpha_i,\ \beta_i \geq 1$, and $(\alpha_i/m_1) + (\beta_i/m_2) < 1$, $i = 1, 2$.

Our work is to consider the existence and attractivity of the maximal periodic solution of the problem (1.1)–(1.3). It should be remarked that our work is not a simple work. The main reason is that the degeneracy of (1.1), (1.2) makes the work of energy estimates more complicated. Since the equations have periodic sources, it is of no meaning to consider the steady state. So, we have to seek a new approach to describe the asymptotic behavior of the nonnegative solutions of the initial boundary value problem. Our idea is to consider all the nonnegative periodic solutions. We fist establish some important estimations on the nonnegative periodic solutions. Then by the De Giorgi iteration technique, we provide a priori estimate of the nonnegative periodic solutions from the upper bound according to the maximum norm. These estimates are crucial for the proof of the existence of the maximal periodic solution and the asymptotic boundedness of the nonnegative solutions of the initial boundary value problem.

This paper is organized as follows. In Section 2, we introduce some necessary preliminaries and the statement of our main results. In Section 3, we give the proof of our main results.

2. Preliminary

In this section, as preliminaries, we present the definition of weak solutions and some useful principles. Since (1.1) and (1.2) are degenerated whenever $u = v = 0$, we focus our main efforts on the discussion of weak solutions.

Definition 2.1. A vector-valued function (u,v) is called to be a weak supsolution to the problem (1.1)–(1.4) in $Q_\tau = \Omega \times (0,\tau)$ with $\tau > 0$ if $|\nabla u^{m_1}|$, $|\nabla v^{m_2}| \in L^2(Q_\tau)$, and for any nonnegative function $\varphi \in C^1(\overline{Q}_\tau)$ with $\varphi|_{\partial\Omega\times[0,\tau)} = 0$ one has

$$\begin{gathered}
\int_\Omega u(x,\tau)\varphi(x,\tau)dx - \int_\Omega u_0(x)\varphi(x,0)dx - \iint_{Q_\tau} u\frac{\partial\varphi}{\partial t}dxdt \\
+ \iint_{Q_\tau} \nabla u^{m_1}\nabla\varphi\, dx\, dt \geq \iint_{Q_\tau} b_1 u^{\alpha_1} v^{\beta_1}\varphi\, dx\, dt, \\
\int_\Omega v(x,\tau)\varphi(x,\tau)dx - \int_\Omega v_0(x)\varphi(x,0)dx - \iint_{Q_\tau} v\frac{\partial\varphi}{\partial t}dx\, dt \\
+ \iint_{Q_\tau} \nabla v^{m_2}\nabla\varphi\, dx\, dt \geq \iint_{Q_\tau} b_2 u^{\alpha_2} v^{\beta_2}\varphi\, dx\, dt, \\
u(x,t) \geq 0, \quad v(x,t) \geq 0, \quad (x,t) \in \partial\Omega \times (0,\tau), \\
u(x,0) \geq u_0(x), \quad v(x,0) \geq v_0(x), \quad x \in \Omega.
\end{gathered} \tag{2.1}$$

Replacing "≥" by "≤" in the above inequalities follows the definition of a weak subsolution. Furthermore, if (u,v) is a weak supersolution as well as a weak subsolution, then we call it a weak solution of the problem (1.1)–(1.4).

Definition 2.2. A vector-valued function (u,v) is said to be a T-periodic solution of the problem (1.1)–(1.3) if it is a solution such that

$$u(\cdot,0) = u(\cdot,T), \quad v(\cdot,0) = v(\cdot,T) \quad \text{a.e in } \Omega. \tag{2.2}$$

A vector-valued function $(\overline{u},\overline{v})$ is said to be a T-periodic supersolution of the problem (1.1)–(1.3) if it is a supersolution such that

$$\overline{u}(\cdot,0) \geq \overline{u}(\cdot,T), \quad \overline{v}(\cdot,0) \geq \overline{v}(\cdot,T) \quad \text{a.e in } \Omega. \tag{2.3}$$

A vector-valued function $(\underline{u},\underline{v})$ is said to be a T-periodic subsolution of the problem (1.1)–(1.3) if it is a subsolution such that

$$\underline{u}(\cdot,0) \leq \underline{u}(\cdot,T), \quad \underline{v}(\cdot,0) \leq \underline{v}(\cdot,T) \quad \text{a.e in } \Omega. \tag{2.4}$$

A pair of supersolution $(\overline{u},\overline{v})$ and subsolution $(\underline{u},\underline{v})$ are called to be ordered if

$$\overline{u} \geq \underline{u}, \quad \overline{v} \geq \underline{v} \quad \text{a.e. in } \overline{Q}_T = \overline{\Omega} \times (0,T). \tag{2.5}$$

Several properties of solutions of problem (1.1)–(1.4) are needed in this paper.

Lemma 2.3 (see [17]). *If $\alpha_i \geq 1$, $\beta_i \geq 1$, $(\alpha_i/m_1) + (\beta_i/m_2) < 1$ with $|\Omega| < M_0$ and M_0 is a constant depending on m_i, α_i, β_i, $i = 1, 2$, then there exist global weak solutions to* (1.1)–(1.4).

Lemma 2.4 (see [20]). *Letting* $(\underline{u}, \underline{v})$ *be a subsolution of the problem* (1.1)–(1.4) *with the initial value* $(\underline{u}_0, \underline{v}_0)$*, and letting* $(\overline{u}, \overline{v})$ *be a supsolution of the problem* (1.1)–(1.4) *with the initial value* $(\overline{u}_0, \overline{v}_0)$*, then* $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ *a.e. in* Q_T *if* $\underline{u}_0 \le \overline{u}_0$, $\underline{v}_0 \le \overline{v}_0$ *a.e. in* Ω.

Lemma 2.5 (regularity [23]). *Let* $u(x,t)$ *be a weak solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m + f(x,t), \quad m > 1, \tag{2.6}$$

subject to the homogeneous Dirichlet condition (1.3). *If* $f \in L^\infty(Q_T)$*, then there exist positive constants* K *and* $\beta \in (0,1)$ *depending only upon* $\tau \in (0,T)$ *and* $\|f\|_\infty$ *such that for any* $(x_1,t_1),(x_2,t_2) \in \overline{\Omega} \times [\tau, T]$*, one has*

$$|u(x_1,t_1) - u(x_2,t_2)| \le K\left(|x_1 - x_2|^\beta + |t_1 - t_2|^{\beta/2}\right). \tag{2.7}$$

The main result of this paper is the following theorem.

Theorem 2.6. *If* $m_i > 1$, $\alpha_i \ge 1$, $\beta_i \ge 1$*, and* $(\alpha_i/m_1) + (\beta_i/m_2) < 1$ *with* $|\Omega| < M_0$ *and* M_0 *is a constant depending on* m_i, α_i, β_i, $i = 1,2$*, then problem* (1.1)–(1.3) *has a maximal periodic solution* (U,V) *which is positive in* Ω^+*. Moreover, if* (u,v) *is the solution of the initial boundary value problem* (1.1)–(1.4) *with nonnegative initial value* (u_0,v_0)*, then for any* $\varepsilon > 0$*, there exists* t_1 *depending on* u_0 *and* ε*,* t_2 *depending on* v_0 *and* ε*, such that*

$$\begin{aligned} &0 \le u \le U + \varepsilon, \quad \text{for } x \in \Omega,\ t \ge t_1, \\ &0 \le v \le V + \varepsilon, \quad \text{for } x \in \Omega,\ t \ge t_2. \end{aligned} \tag{2.8}$$

3. The Main Results

In this section, we first show some important estimates on the solutions of the periodic problem (1.1)–(1.3). Then, by the De Giorgi iteration technique, we establish the a prior upper bound of periodic solutions of (1.1)–(1.3), which is used to show the existence of the maximal periodic solution of (1.1)–(1.3) and its attractivity with respective to the nonnegative solutions of the initial boundary value problem (1.1)–(1.4).

Lemma 3.1. *Let* (u,v) *be nonnegative solution of* (1.1)–(1.3). *If* $\alpha_i \ge 1$, $\beta_i \ge 1$, $(\alpha_i/m_1)+(\beta_i/m_2) < 1$ *with* $|\Omega| < M_0$ *and* M_0 *is a constant depending on* m_i, α_i, β_i, $i = 1,2$*, then there exists positive constants* r *and* s *large enough such that*

$$\frac{\alpha_2}{m_2 - \beta_2} < \frac{m_1 + r - 1}{m_2 + s - 1} < \frac{m_1 - \alpha_1}{\beta_1}, \tag{3.1}$$

$$\|u\|_{L^r(Q_T)} \le C, \qquad \|v\|_{L^s(Q_T)} \le C, \tag{3.2}$$

where $C > 0$ *is a positive constant depending on* m_1, m_2, α_1, α_2, β_1, β_2, r, s*, and* $|\Omega|$.

Proof. For $r > 1$, multiplying (1.1) by u^{r-1} and integrating over Q_T, by the periodic boundary value condition, we have

$$\frac{4(r-1)m_1}{(m_1+r-1)^2}\int_{\Omega}\left|\nabla u^{(m_1+r-1)/2}\right|^2 dx\,dt = \iint_{Q_T} b_1(x,t)u^{\alpha_1+r-1}v^{\beta_1}\,dx\,dt, \tag{3.3}$$

that is,

$$\int_{\Omega}\left|\nabla u^{(m_1+r-1)/2}\right|^2 dx\,dt \leq \frac{C_b(m_1+r-1)^2}{4(r-1)m_1}\iint_{Q_T} u^{\alpha_1+r-1}v^{\beta_1}\,dx\,dt, \tag{3.4}$$

where $C_b = \underset{Q_T}{b_1(x,t)}$. By the Poincaré inequality, we have

$$\int_{\Omega} u_{\varepsilon}^{m_1+r-1}dx \leq C\int_{\Omega}\left|\nabla u_{\varepsilon}^{(m_1+r-1)/2}\right|^2 dx, \tag{3.5}$$

where C is a constant depending only on $|\Omega|$ and N. Notice that $(\alpha_1/m_1) + (\beta_1/m_2) < 1$ implies $\alpha_1 < m_1$. Furthermore, we have $\alpha_1 + r - 1 < m_1 + r - 1$. Then, by Young's inequality, we obtain

$$u^{\alpha_1+r-1}v^{\beta_1} \leq \frac{1}{2}\frac{(r-1)m_1}{CC_b}\left(\frac{2}{m_1+r-1}\right)^2 u^{m_1+r-1} + C_1 v^{\beta_1(m_1+r-1)/(m_1-\alpha_1)}, \tag{3.6}$$

where C_1 is the constant of Young's inequality. Then, from (3.4), we have

$$\iint_{Q_T} u^{m_1+r-1}dx\,dt \leq \frac{1}{2}\iint_{Q_T} u^{m_1+r-1}dx\,dt + C_1\iint_{Q_T} v^{\beta_1(m_1+r-1)/(m_1-\alpha_1)}dx\,dt, \tag{3.7}$$

that is,

$$\iint_{Q_T} u^{m_1+r-1}dx\,dt \leq C_1\iint_{Q_T} v^{\beta_1(m_1+r-1)/(m_1-\alpha_1)}dx\,dt. \tag{3.8}$$

Similarly, we get an estimate for v^s with $s > 1$, that is,

$$\iint_{Q_T} v^{m_2+s-1}dx\,dt \leq C_2\iint_{Q_T} u^{\alpha_2(m_2+s-1)/(m_2-\beta_2)}dx\,dt. \tag{3.9}$$

Hence,

$$\begin{aligned}&\iint_{Q_T} u^{m_1+r-1}dx\,dt + \iint_{Q_T} v^{m_2+s-1}dx\,dt\\ &\quad\leq C_1\iint_{Q_T} v^{\beta_1(m_1+r-1)/(m_1-\alpha_1)}dx\,dt + C_2\iint_{Q_T} u^{\alpha_2(m_2+s-1)/(m_2-\beta_2)}dx\,dt.\end{aligned} \tag{3.10}$$

Notice that, $(\alpha_i/m_1)+(\beta_i/m_2)<1$, $i=1,2$, implies $\alpha_2\beta_1<(m_1-\alpha_1)(m_2-\beta_2)$. Then there exist $r\geq\max\{2(m_1+\alpha_1),2\alpha_2\}$ and $s\geq\max\{2(m_2+\beta_2),2\beta_1\}$ such that

$$\frac{\beta_1}{m_1-\alpha_1}<\frac{m_2+s-1}{m_1+r-1}<\frac{m_2-\beta_2}{\alpha_2}. \tag{3.11}$$

By Young's inequality, we have

$$\begin{aligned}\iint_{Q_T} u^{\alpha_2(m_2+s-1)/(m_2-\beta_2)}dx\,dt &\leq \frac{1}{2C_2}\iint_{Q_T} u^{m_1+r-1}dx\,dt+C|Q_T|,\\ \iint_{Q_T} v^{\beta_1(m_1+r-1)/(m_1-\alpha_1)}dx\,dt &\leq \frac{1}{2C_1}\iint_{Q_T} v^{m_2+s-1}dx\,dt+C|Q_T|.\end{aligned} \tag{3.12}$$

Together with (3.10), we obtain

$$\iint_{Q_T} u^{m_1+r-1}dx\,dt+\iint_{Q_T} v^{m_2+s-1}dx\,dt\leq C. \tag{3.13}$$

Thus, we prove the inequality (3.2). □

Lemma 3.2. *Let (u,v) be nonnegative solution of (1.1)–(1.3). If $\alpha_i\geq 1$, $\beta_i\geq 1$, $(\alpha_i/m_1)+(\beta_i/m_2)<1$ with $|\Omega|<M_0$ and M_0 is a constant depending on m_i,α_i,β_i, $i=1,2$, then one has*

$$\iint_{Q_T}|\nabla u^{m_1}|^2dx\,dt\leq C,\qquad \iint_{Q_T}|\nabla v^{m_2}|^2dx\,dt\leq C, \tag{3.14}$$

where $C>0$ is a positive constant depending on m_1, m_2, α_1, α_2, β_1, β_2, r, s, and $|\Omega|$.

Proof. Multiplying (1.1) by u^{m_1} and integrating over Q_T, by Hölder's equality, we have

$$\begin{aligned}\iint_{Q_T}|\nabla u^{m_1}|^2dx\,dt &\leq \iint_{Q_T} u^{\alpha_1+m_1}v^{\beta_1}dx\,dt\\ &\leq \left(\iint_{Q_T}u^{2(\alpha_1+m_1)}dx\,dt\right)^{1/2}\left(\iint_{Q_T}v^{2\beta_1}dx\,dt\right)^{1/2}.\end{aligned} \tag{3.15}$$

Taking $r\geq\max\{2(\alpha_1+m_1),2\beta_2\}$, $s\geq\max\{2(\beta_2+m_2),2\alpha_1\}$, by Lemma 3.1, we can obtain the first inequality in (3.14). The same is true for the second inequality in (3.14). □

Before we show the uniform super bound of maximum modulus, we first introduce a lemma as follows (see [24]).

Lemma 3.3. *Suppose that a sequence y_h, $h=0,1,2,\dots$ of nonnegative numbers satisfies the recursion relation*

$$y_{h+1}\leq cb^hy_h^{1+\varepsilon},\quad h=0,1,\dots, \tag{3.16}$$

with some positive constants c, ε and $b \geq 1$. Then,

$$y_h \leq c^{((1+\varepsilon)^h-1)/\varepsilon} b^{((1+\varepsilon)^h-1)/\varepsilon^2 - h/\varepsilon} y_0^{(1+\varepsilon)^h}. \tag{3.17}$$

In particular, if

$$y_0 \leq \theta = c^{-1/\varepsilon} b^{-1/\varepsilon^2}, \quad b > 1, \tag{3.18}$$

then,

$$y_h \leq \theta b^{-h/\varepsilon}, \tag{3.19}$$

and consequently $y_h \to 0$ for $h \to \infty$.

Lemma 3.4. *Let (u, v) be a solution of (1.1)–(1.3). If $\alpha_i \geq 1$, $\beta_i \geq 1$, $(\alpha_i/m_1) + (\beta_i/m_2) < 1$ with $|\Omega| < M_0$ and M_0 is a constant depending on m_i, α_i, β_i, $i = 1, 2$, then there is a positive constant C such that*

$$\|u\|_{L^\infty(Q_T)} \leq C, \qquad \|v\|_{L^\infty(Q_T)} \leq C. \tag{3.20}$$

Proof. Let k be a positive constant. Multiplying (1.1) by $(u-k)_+^{m_1}$ and integrating over Q_T, we have

$$\begin{aligned} \frac{1}{m_1+1} \iint_{Q_T} \frac{\partial}{\partial t}(u-k)_+^{m_1+1} dx\,dt &+ \iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dx\,dt \\ &= \iint_{Q_T} b_1(x,t) u^{\alpha_1} v^{\beta_1} (u-k)_+^{m_1} dx\,dt, \end{aligned} \tag{3.21}$$

where $s_+ = \max\{s, 0\}$. Denote that $\mu(k) = \operatorname{mes}\{(x,t) \in Q_T : u(x,t) > k\}$. By Lemma 3.1 (with r and s large enough) and Hölder's inequality, we have

$$\begin{aligned} \frac{1}{m_1+1} \iint_{Q_T} \frac{\partial}{\partial t}(u-k)_+^{m_1+1} dx\,dt &+ \iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dx\,dt \\ &\leq C\left(\iint_{Q_T} \left(u^{\alpha_1} v^{\beta_1}\right)^{\xi'} dx\,dt\right)^{\xi'} \left(\iint_{Q_T} (u-k)_+^{m_1\xi} dx\,dt\right)^{1/\xi} \\ &\leq C\left(\iint_{Q_T} (u-k)_+^{m_1\xi\eta} dx\,dt\right)^{1/\xi\eta} \mu(k)^{(1-1/\eta)(1/\xi)}, \end{aligned} \tag{3.22}$$

where $\xi, \eta > 1$ are to be determined. Using the Nirenberg-Gagliardo inequality with Lemma 3.1, we have

$$\left(\iint_{Q_T} (u-k)_+^{m_1\xi\eta} dx\, dt\right)^{1/\xi\eta} \leq C\left(\iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dx\, dt\right)^{\theta/2}, \tag{3.23}$$

where

$$\theta = \left(1-\frac{1}{\xi\eta}\right)\left(\frac{1}{N}-\frac{1}{2}+1\right)^{-1} \in (0,1). \tag{3.24}$$

Substituting (3.22) and (3.23) in (3.21), we have

$$\iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dxdt \leq C\left(\iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dx\, dt\right)^{\theta/2} \mu(k)^{(1-1/\eta)(1/\xi)}. \tag{3.25}$$

Setting

$$w(k) = \iint_{Q_T} \left|\nabla (u-k)_+^{m_1}\right|^2 dx\, dt, \tag{3.26}$$

from (3.25) we obtain

$$w(k) \leq C\mu(k)^{(2/(2-\theta))(1-1/\eta)(1/\xi)}. \tag{3.27}$$

Take $k_h = M(2-2^{-h})$, $h = 0, 1, \ldots$, and $M > 0$ is to be determined. Then, we have

$$(k_{h+1}-k_h)^{m_1\xi\eta}\mu(k_{h+1}) \leq \iint_{Q_T} (u-k_h)_+^{m_1\xi\eta} dx\, dt \leq Cw(k_h)^{\xi\eta\theta/2}. \tag{3.28}$$

From (3.26), we have

$$\mu(k_{h+1}) \leq C2^{hm_1\xi\eta}\mu(k_h)^{\theta(\eta-1)/(2-\theta)} = Cb^h\mu(k_h)^\gamma, \tag{3.29}$$

where $b = 2^{m_1\xi\eta}$ and $\gamma = (\eta-1)(\xi\eta-1)N/(2\xi\eta+N)$. For any constant $\xi > 1$, take η to be a positive constant satisfying

$$\eta > \max\left\{2, \frac{2\xi+N}{\xi N}-1\right\}, \tag{3.30}$$

then we have $\gamma > 1$. By Lemma 3.1, we can select M large enough such that

$$\mu(k_0) = \mu(M) \leq C^{-1/(\gamma-1)}4^{-1/(\gamma-1)^2}. \tag{3.31}$$

According to Lemma 3.3, we have $\mu(k_h) \to 0$, as $h \to +\infty$, which implies that $u(x,t) \le 2M$ in Q_T. The uniform estimate for $\|v(x,t)\|_{L^\infty(Q_T)}$ may be obtained by a similar method. The proof is completed. □

Let μ, ψ be the first eigenvalue and its corresponding eigenfunction to the Laplacian operator $-\Delta$ on some domain $\Omega' \supset\supset \Omega$ with respect to homogeneous Dirichlet data. It is clear that $\psi_{(x)} > 0$ for all $x \in \overline{\Omega}$.

Now we give the proof of the main results of this paper.

Proof of Theorem 2.6. We first establish the existence of the maximal periodic solution $(U(x,t), V(x,t))$ of the problem (1.1)–(1.3). Define the Poincaré mapping

$$\begin{gathered} T = (T_1, T_2) : C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \longrightarrow C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right), \\ T(u_0(x), v_0(x)) = (u(x,T), v(x,T)), \end{gathered} \tag{3.32}$$

where $(u(x,t), v(x,t))$ is the solution of the initial boundary value problem (1.1)–(1.4) with initial value $(u_0(x), v_0(x))$. A similar argument as that in [22] shows that the map T is well defined.

Let $(u_n(x,t), v_n(x,t))$ be the solution of the problem (1.1)–(1.4) with initial value

$$(u_0(x), v_0(x)) = (\overline{u}(x), \overline{v}(x)) = \left(K_1\varphi_1, K_2\varphi_2\right), \tag{3.33}$$

where K_1, K_2, φ_1, and φ_2 are taken as those in [22]. Then, by comparison principle, we have

$$\begin{gathered} (u_n(x,T), v_n(x,T)) = T^n(\overline{u}(x), \overline{v}(x)), \\ u_{n+1}(x,t) \le u_n(x,t) \le \overline{u}(x), \qquad v_{n+1}(x,t) \le v_n(x,t) \le \overline{v}(x). \end{gathered} \tag{3.34}$$

A standard argument shows that there exist $(u^*(x), v^*(x)) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and a subsequence of $\{T^n(\overline{u}(x))\}$, denoted by itself for simplicity, such that

$$(u^*(x), v^*(x)) = \lim_{n\to\infty} T^n(\overline{u}(x), \overline{v}(x)). \tag{3.35}$$

Similar to the proof of Theorem 4.1 in [25], we can prove that $(U(x,t), V(x,t))$, which is the even extension of the solution of the initial boundary value (1.1)–(1.4) with the initial value $(u^*(x), v^*(x))$, is a periodic solution of (1.1)–(1.3). For any nonnegative periodic solution $(u(x,t), v(x,t))$ of (1.1)–(1.3), by Lemma 3.4, we have

$$u(x,t) \le C_0, \quad v(x,t) \le C_0 \quad \text{for } (x,t) \in Q_T. \tag{3.36}$$

Taking

$$K_1 \ge \frac{C_0}{\min_{x\in\Omega} \varphi_1^{1/m_1}(x)}, \qquad K_2 \ge \frac{C_0}{\min_{x\in\Omega} \varphi_2^{1/m_2}(x)}, \tag{3.37}$$

to be combined with the comparison principle and $u^*(x) \geq u(x,0), v^*(x) \geq v(x,0)$, then we obtain $U(x,t) \geq u(x,t)$, $V(x,t) \geq v(x,t)$, which implies that $(U(x,t), V(x,t))$ is the maximal periodic solution of (1.1)–(1.3).

For any given nonnegative initial value $(u_0(x), v_0(x))$, let $(u(x,t), v(x,t))$ be the solution of the initial boundary problem (1.1)–(1.4), and let $(\omega_1(x,t), \omega_2(x,t))$ be the solution of (1.1)–(1.4) with initial value $(\omega_1(x,0), \omega_2(x,0)) = (R_1\varphi_1(x), R_2\varphi_2(x))$, where R_1, R_2 satisfy the same conditions as K_1, K_2 and

$$R_1 \geq \frac{\|u_0\|_{L^\infty}}{\min_{x\in\Omega} \varphi_1^{1/m_1}(x)}, \qquad R_2 \geq \frac{\|v_0\|_{L^\infty}}{\min_{x\in\Omega} \varphi_2^{1/m_2}(x)}. \tag{3.38}$$

For any $(x,t) \in Q_T$, $k = 0, 1, 2, \ldots$, we have

$$u(x, t + kT) \leq w_1(x, t + kT), \qquad v(x, t + kT) \leq w_2(x, t + kT). \tag{3.39}$$

A similar argument as that in [25] shows that

$$\left(\omega_1^*(x,t), \omega_2^*(x,t)\right) = \left(\lim_{k\to\infty} \omega_1(x, t + kT), \lim_{k\to\infty} \omega_2(x, t + kT)\right), \tag{3.40}$$

and $(\omega_1^*(x,t), \omega_2^*(x,t))$ is a nontrivial nonnegative periodic solution of (1.1)–(1.3). Therefore, for any $\varepsilon > 0$, there exists k_0 such that

$$\begin{aligned} u(x, t + kT) &\leq \omega_1^*(x,t) + \varepsilon \leq U(x,t) + \varepsilon, \\ v(x, t + kT) &\leq \omega_2^*(x,t) + \varepsilon \leq V(x,t) + \varepsilon, \end{aligned} \tag{3.41}$$

for any $k \geq k_0$ and $(x,t) \in \overline{Q}_T$. Taking the periodicity of $\omega_1^*(x,t)$, $\omega_2^*(x,t)$, $U(x,t)$, and $V(x,t)$ into account, the proof of the theorem is completed. □

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